

AN EXAMPLE ON DEFORMATION OF A PAIR

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Abstract. In this paper we give an example to show Clemens' conjecture is not a first order deformation problem.

To give a rigorous explanation of the abstract, we start with Clemens' conjecture. Let X_0 be a smooth quintic of \mathbf{P}^4 over \mathbb{C} , and C_0 be an irreducible rational curve of degree d . In [1], its original 1986 statement, Clemens proposed:

(1) the generic quintic threefold X_0 admits only finitely many rational curves C_0 of each degree.

(2) Each rational curve C_0 is a smoothly embedded \mathbf{P}^1 with normal bundle

$$(0.1) \quad \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

(3) All the rational curves on X_0 are mutually disjoint. The number of rational curves of degree d on X_0 is

$$(0.2) \quad (\text{interesting number}) \cdot 5^3 \cdot d.$$

In [2], we proved

THEOREM 0.1. *Let $X_0 \subset \mathbf{P}^4$ be a generic quintic threefold. For each degree $d \geq 1$,*

- (i) *there are only finitely many irreducible rational curves $C_0 \subset X_0$ of degree d .*
- (ii) *Each rational curve in (i) is an immersed rational curve with normal bundle*

$$N_{C_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

By “immersed rational curve” we mean that the normalization map is an immersion.

Our proof uses higher order deformations of the pair, i.e. we used the genericity of the quintic X_0 . This paper concerns this assumption of the conjecture, i.e. X_0 is generic. The question is

In Clemens' conjecture, can the “genericity” of the quintic X_0 be replaced by some other more specific condition on the pair?

We are far from being ready to even formulate any conjecture on the question. This paper gives a negative answer to a specific conjectural condition. Let's start with this first order condition. Let $H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ denote the vector space of homogeneous polynomials of degree $5 = \deg(X_0)$ in 5 variables. Let $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ such that

$$X_0 = \text{div}(f_0)$$

is a smooth hypersurface. Let

$$[f_0] \in S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$$

denote the corresponding point of f_0 in the projectivization. Let

$$(0.3) \quad M_d = H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5}.$$

So the normalization of C_0 , $c_0 : \mathbf{P}^1 \rightarrow C_0$ lies in the projectivization of M_d . Let

$$(0.4) \quad \Gamma$$

be the incidence scheme

$$(0.5) \quad \{(c, [f]) : f(c(t)) = 0\} \subset M_d \times S$$

containing the point $(c_0, [f_0])$.

Condition 0.1. : *The projection*

$$(0.6) \quad T_{(c_0, f_0)}\Gamma \xrightarrow{\pi} T_{[f_0]}S$$

is surjective.

Motivation for the condition 0.1:

(1) The main motivation is the question above. Can the genericity of the quintic be replaced by condition 0.1 ? There is an evidence for an affirmative answer. Condition 0.1 was used in [3], [4]. In particular it is shown in [3] that the condition 0.1 is responsible for the global generation of the twisted normal bundle $N_{C_0/X_0}(d)$. So it is natural to conjecture it may also be responsible for the vanishing of H^1 of the un-twisted normal bundle N_{C_0/X_0} .

(3) If X_0 is generic, the condition 0.1 is automatic. Thus the studying only the first orders in Clemens' conjecture is equivalent to assuming the condition 0.1 and removing the genericity condition of the quintic.

However the following theorem shows that expectations above are wrong.

THEOREM 0.2. *There are a smooth quintic threefold X_0 and a plane rational curve $C_0 \subset X_0$ of degree 3 with the normalization $c_0 : \mathbf{P}^1 \rightarrow C_0$, such that the condition 0.1 is satisfied, i.e. the projection*

$$(0.7) \quad T_{(c_0, f_0)}\Gamma \xrightarrow{\pi} T_{[f_0]}S$$

is surjective, but $H^1(N_{c_0/X_0}) \neq 0$, where N_{c_0/X_0} is defined to be the normal sheaf of the normalization map c_0 .

Remark Actually it is not so difficult to see that for a hypersurface X_0 of another types (such that Fano) the condition 0.1 is insufficient for the vanishing of H^1 of the normal sheaf. But this becomes non-trivial for a quintic. It was proved in [3] that the condition 0.1 indeed implies the Clemens' conjecture for lines. We believe that for rational curves of degree less than 3, the condition 0.1 should be sufficient for Clemens' conjecture. Thus 3 is expected to be the minimum degree for the condition 0.1 to be insufficient for Clemens' conjecture.

Proof. Since $c_0 : \mathbf{P}^1 \rightarrow \mathbf{P}^4$ generically is an embedding, the normal sheaves

$$(0.8) \quad N_{c_0/\mathbf{P}^4}, N_{c_0/X_0}$$

are well-defined and there is an exact sequence of sheaves over \mathbf{P}^1 ,

$$(0.9) \quad 0 \rightarrow N_{c_0/X_0} \rightarrow N_{c_0/\mathbf{P}^4} \rightarrow c_0^*(N_{X_0/\mathbf{P}^4}) \rightarrow 0.$$

This induces an exact sequence of vector spaces,

$$(0.10) \quad H^0(N_{c_0/\mathbf{P}^4}) \xrightarrow{\nu^s} H^0(c_0^*(N_{X_0/\mathbf{P}^4})) \rightarrow H^1(N_{c_0/X_0}) \rightarrow 0.$$

We need to add a few more words about ν^s for the calculation later. In [2], we proved that if c_0 is a birational to its image, there is a natural surjection

$$(0.11) \quad H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5} \rightarrow H^0(c_0^*(T_{\mathbf{P}^1}))$$

whose kernel is a one-dimensional line through the origin. Hence we obtain a composition

$$(0.12) \quad \begin{array}{ccc} \phi : H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5} & \xrightarrow{\phi} & H^0(c_0^*(N_{X_0/\mathbf{P}^4})) \\ (\alpha_0, \alpha_4) & \rightarrow & \sum_{i=0}^4 \frac{\partial c_0^*(f_0)}{\partial \alpha_i}. \end{array}$$

(The partial derivative $\frac{\partial c_0^*(f_0)}{\partial \alpha_i}$ denotes $\frac{\partial c^*(f_0)}{\partial \alpha_i}|_{c_0}$). In (0.12), we identify

$$(0.13) \quad T_0(H^0(\mathcal{O}_{\mathbf{P}^1}(d)))$$

with

$$(0.14) \quad H^0(\mathcal{O}_{\mathbf{P}^1}(d)).$$

by the linearity of $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$. Furthermore ν^s is surjective if and only if ϕ is surjective which is equivalent to the kernel of ϕ being the tangent space $T_I(GL(2))$ of the orbit by the $GL(2)$ action on

$$H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5}$$

induced from the automorphisms of \mathbb{C}^2 for $\mathbf{P}(\mathbb{C}^2) = \mathbf{P}^1$.

On the other hand, there is a linear map μ at c_0 , defined by

$$(0.15) \quad \begin{array}{ccc} H^0(\mathcal{O}_{\mathbf{P}^4}(5)) & \xrightarrow{\mu} & H^0(c_0^*(N_{X_0/\mathbf{P}^4})) \\ f & \rightarrow & c_0^*(f). \end{array}$$

The differential μ^s of it is also a linear map

$$\begin{array}{ccc} T_{f_0} H^0(\mathcal{O}_{\mathbf{P}^4}(5)) = H^0(\mathcal{O}_{\mathbf{P}^4}(5)) & \xrightarrow{\mu^s} & T_0 H^0(c_0^*(N_{X_0/\mathbf{P}^4})) = H^0(c_0^*(N_{X_0/\mathbf{P}^4})) \\ \alpha & \rightarrow & \frac{\partial c_0^*(f_0)}{\partial \alpha}. \end{array}$$

Because of (0.10), $H^1(N_{c_0/X_0}) = 0$ is equivalent to the surjectivity of ν^s . At the meantime, the condition 0.1 can be expressed as

$$(0.16) \quad \text{image}(\mu^s) \subset \text{image}(\nu^s).$$

Therefore we would like to construct an example such that (0.16) holds but ν^s is not surjective.

The following is this example. Let z_0, \dots, z_4 be the homogeneous coordinates of \mathbf{P}^4 . Let $V \subset \mathbf{P}^4$ be the plane

$$(0.17) \quad \{[z_0 \cdots, z_4] : z_0 = z_1 = 0\}.$$

There is a natural embedding

$$(0.18) \quad H^0(\mathcal{O}_V(n)) \xrightarrow{j} H^0(\mathcal{O}_{\mathbf{P}^4}(n)).$$

Let C_0 be a generic cubic rational curve on V , i.e. C_0 is defined by a polynomial

$$(0.19) \quad g_2 \in H^0(\mathcal{O}_V(3))$$

that has exactly one nodal singularity. Let $c_0 : \mathbf{P}^1 \rightarrow C_0$ be its normalization. Let $g_0 \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$, $q \in H^0(\mathcal{O}_V(2))$ such that

$$(0.20) \quad \xi^*(g_0)$$

and $c_0^*(q)$ have at least three common zeros, where ξ is the embedding

$$\mathbf{P}^1 \hookrightarrow C_0 \hookrightarrow V \hookrightarrow \mathbf{P}^4.$$

Let

$$(0.21) \quad f_0 = z_0 g_0 + z_1 g_1 + g_2](q)$$

where $g_1 \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ is generic, and all other polynomials g_0, g_2, q are generic in the chosen types above.

Then we can check that

- (1) $X_0 = \text{div}(f_0)$ is a smooth quintic threefold,
- (2) $C_0 = \text{im}(c_0)$ is a singular, cubic plane rational curve. But it is a global immersion,
- (3) C_0 lies on X_0 .

In the following calculation, we identify

$$(0.22) \quad T_0(H^0(\mathcal{O}_{\mathbf{P}^1}(m)))$$

with

$$(0.23) \quad H^0(\mathcal{O}_{\mathbf{P}^1}(m)).$$

by the linearity of $H^0(\mathcal{O}_{\mathbf{P}^1}(m))$.

Consider the linear map λ

$$(0.24) \quad \begin{array}{ccc} H^0(c_0^*(\mathcal{O}_V(1)))^{\oplus 5} & \xrightarrow{\lambda} & H^0(c_0^*(\mathcal{O}_V(5))) \\ c_0^*(\alpha_0, \dots, \alpha_4) & \rightarrow & c_0^*(\sum_{i=0}^1 \frac{\partial g_i|_V}{\partial \alpha_i} + q \sum_{i=2}^4 \frac{\partial g_2}{\partial \alpha_i}). \end{array}$$

Because of the genericity of g_0, g_1, g_2 , the linear map λ is surjective. This shows that the condition 0.1 is satisfied.

Because

$$(0.25) \quad \xi^*(f_0)$$

and $c_0^*(q)$ have at least three common zeros, we find non-zero

$$\beta_0, \beta_2, \beta_3, \beta_4 \in H^0(\mathcal{O}_{\mathbf{P}^1}(3))$$

such that

$$(\beta_0, 0, \beta_2, \beta_3, \beta_4)$$

is not in the tangent space $T_I(GL(2))$ of the automorphism $GL(2)$ of

$$(0.26) \quad H^0(\mathcal{O}_{\mathbf{P}^1}(3))^{\oplus 5}$$

(because β_0 is not identically zero) and

$$(0.27) \quad \beta_0 \cdot c_0^*(g_0) + \xi^*(q) \cdot \sum_{i=2}^4 \beta_i \cdot \frac{\partial g_2}{\partial z_i} = 0.$$

Hence

$$(\beta_0, 0, \beta_2, \beta_3, \beta_4)$$

is in the kernel of ϕ but not in the $T_I(GL(2))$. Then

$$(0.28) \quad \dim(\ker(\phi)) \geq 5.$$

This shows ν^s for this pair X_0, C_0 is not surjective.

The theorem is proved.

□

This example offers a scenario that

$$(0.29) \quad \text{image}(\mu^s) = \text{image}(\nu^s) \subsetneq H^0(\mathcal{O}_{\mathbf{P}^1}(15)).$$

REFERENCES

- [1] H. CLEMENS , *Curves on higher-dimensional complex projective manifolds*, Proc. International Cong.Math., Berkeley 1986, pp. 634–640.
- [2] B. WANG , *Rational curves on generic quintic threefolds*, arXiv:1202.2831, 2014
- [3] ———, *First Order Deformations of Pairs of a Rational Curve and a Hypersurface*, Asian J. Math. 18 (2014), 101-116.
- [4] ———, *First order deformation of pairs and non-existence of rational curves*, To appear in Rocky Mountain Journal of Mathematics (2014)